A Modular Proof of Semantic Completeness for Normal Systems beyond the Modal Cube, Formalised in HOLMS

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Abstract

We communicate here the most recent extension of HOLMS, our library for modal logics aimed at introducing automated modal reasoning within the HOL Light proof assistant. Based on a uniform proof strategy, we present a more refined formal proof of completeness for systems within *and beyond* the S5-normal modal cube, notably Gödel-Löb logic. We report on our development by adopting a measure of its modularity based on Strachey's distinction between parametric and ad hoc polymorphic code.

Keywords

Modal logic, HOL Light, Completeness theorems, Interactive theorem proving, Proof libraries

1. Introduction

Starting from the 1970s, modal logic has proven to be a central tool in various technical and scientific disciplines, including knowledge representation, formalisation of reasoning under uncertainty, the study of multi-agent systems, the analysis of computational processes, the verification of consistency of normative corpora, and the modelling of decision-making processes [1, 2, 3, 4]. In computer science, different modal operators are also able to capture abstract properties of: computation with side effects [5, 6, 7]; properties of program executions [8, 9]; applicative functors in functional programming [10, 11, 12]; recursive calls in programming languages [13, 14]; and information flow and knowledge dynamics in communication protocols [15, 16, 17, 18].

These and many other examples show that the interest of computer science in modal logic extends beyond the boundaries of the so-called S_5-normal cube, i.e. the class of normal modal logics obtained from the minimal axiomatic system $\mathbb{K}$ by combining the axioms among D, T, B, 4, and 5. The proper inclusions between these systems can be summarised graphically in the cube shown in Figure 1.

For each system in the modal cube, it is possible to provide a proof of adequacy (soundness and completeness) for the standard relational semantics – also known as "possible worlds" or "Kripke" semantics – by applying in a uniform and modular way the canonical model method, followed by appropriate filtration lemmas specific to each system under consideration [19].

However, this uniform and modular approach fails for specific and relevant cube extensions, even when remaining within the classical and normal modal systems class. Notoriously, Gödel-Löb logic

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Axiom schema	Semantic condition
D : $\Box A \to \Diamond A$	Seriality
T : $\Box A \to A$	Reflexivity
B : $A \to \Box\Diamond A$	Symmetry
4 : $\Box A \to \Box\Box A$	Transitivity
5 : $\Diamond A \to \Box\Diamond A$	Euclidianness
GL : $\Box(\Box A \to A) \to \Box A$	Transitivity and Noetherianity

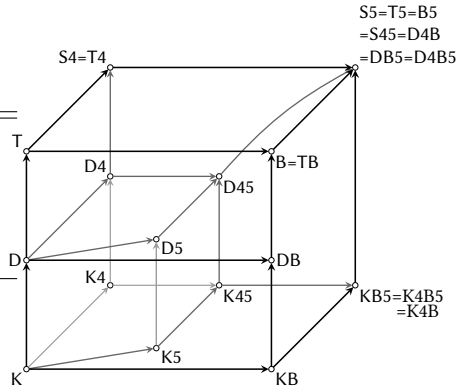

Figure 1: The S_5-normal cube. For any systems S,S', an arrow S $\to$ S′ denotes proper inclusion S $\subsetneq$ S′, i.e., that S' is logically stronger than S.

(GL) is one of the most significant exceptions to the cube: although the logic is complete for standard relational semantics, the proof of this property cannot proceed via a canonical construction–partly due to issues related to incompactness [20, 21]. Nonetheless, it is possible to use a different, yet equally uniform and modular strategy, which subsumes the "canonical model + filtrations" method to obtain adequacy and the finite model property for GL and the systems within the S_5-normal cube, as documented in [22, Ch. 5].

In the present work, we report on the latest update of our implementation of that proof strategy for completeness within the more general HOLMS framework, which currently covers the systems K, T, K4, S4, B, S5 and GL.

1.1. Source code

The latest version of HOLMS is freely accessible from our repository and is archived on Software Heritage. The readme file from the git repository and the project webpage provide some pointers to the main contents of the library.

2. Modular completeness proof, formalised

2.1. HOLMS framework

Our HOLMS framework (HOL Light Library for Modal Systems) [23, 24] has incorporated from its inception a formalisation of the aforementioned adequacy proof with respect to finite models. Initially, this formalisation addressed solely GL [25, 26]. Subsequently, we extended it to encompass three additional normal systems within the cube: K, T, and K4. In the present setting, we present a further extension of it, covering the adequacy proofs and the decision procedures for S4, B and S5.

Our research pursuits with HOLMS are twofold, addressing both practical implementation and theoretical foundations.

From the **practical perspective**, we aim to enhance HOL Light's capabilities in automated modal reasoning. The current version of HOLMS extends the proof assistant's deductive apparatus by introducing a novel inference rule of the proof assistant (`HOLMS_RULE`). This rule determines whether a given input modal formula constitutes a theorem within a specified modal logic selected from K, T, K4, S4, B, S5 and GL. When the formula fails to qualify as a theorem, the rule generates an appropriate countermodel tailored to the established logic.[1] The automation process is grounded in a shallow embedding of root-first proof search within the labelled sequent calculus associated with each logic

[1]The general mechanism provides a semi-decision procedure for K4. However, the literature on labelled sequent calculi offers uniform solutions to this issue [27], which may be incorporated in future versions of HOLMS.

under consideration [28, 29]. Henceforth, in HOLMS we have three interconnected presentations of normal modal logics: (i) axiomatic calculi, (ii) relational semantics (both deeply embedded), and (iii) labelled sequent calculi, which we shallow embed in the goal-stack mechanism of HOL Light as a certificate of correctness of the decision procedure behind `HOLMS_RULE`.

From the **theoretical standpoint**, we intend to develop the library according to a precisely defined modular architecture. Following a compositional implementation methodology, this development generalises our previous GL library for HOL Light. This methodology centres upon the previously referenced scalable and uniform proof of semantic adequacy. To precisely measure the proof modularity – and the same property of the associated formalisation – we implemented a precise coding discipline, inspired by [30]. In developing HOLMS, we have distinguished between: *parametric polymorphic* code, fully independent of specific parameter instantiations; and *ad-hoc polymorphic* code, whose components are tailored to the modal logic under consideration. We use that colour convention consistently throughout the following pages.[2] Figure 2 summarises the modularity of the current version of our code, together with the specific files where each component is implemented. Here, $*$ ranges over the different modal logics considered, that is $* \in \{\mathbb{K}$, $\mathbb{T}$, $\mathbb{K}4$, $\mathbb{S}4$, $\mathbb{B}$, $\mathbb{S}5$, $\mathbb{GL}$ $\}$.

Syntax		`modal.ml`
Semantics		`modal.ml`
Correspondence Theory	Concepts	`parametric_correspondence.ml`
	Lemmata	`ad_hoc_correspondence.ml`
Soundness		`gen_completeness.ml`
Completeness	"Standard" Model	`gen_completeness.ml`
	Truth Lemma	`gen_completeness.ml`
	Countermodel Lemma	`gen_completeness.ml`
	"Standard" Relation	`*_completeness.ml`
	Identification of the "Standard" Model	`*_completeness.ml`
	Type Generalisation	`*_completeness.ml`
Shallow Embedding		`gen_decid.ml` and `*_decid.ml`
Decision Procedure		`gen_decid.ml` and `*_decid.ml`

Figure 2: Measure of the modularity of our implementation

2.2. Syntax and semantics

In our formalisation of modal systems, HOL Light serves as the *metatheory*, while modal logics are treated as *object logics*. This distinction requires the embedding of an object language within HOL Light—to explicitly differentiate between formal statements of the modal language (e.g. $\Box A \to \Box\Box A$) and statements about modal systems in the metatheory (e.g. `|- ∀A. ⊢K4 □A → □□A`).

First, we define a grammar for modal formulas by an inductive definition . Next, we uniformly introduce axiomatic proof systems. To avoid code duplication in the embeddings of Hilbert calculi for normal modal logics, we define a ternary deducibility relation $\mathcal{S}.\mathcal{H} \vdash A$ that is parametric to a set of axiom schemata $\mathcal{S}$ and considers a set of hypotheses $\mathcal{H}$. Such a formal predicate conceptualises the abstract notion of deducibility close to the one discussed in [31], based on a minimal deductive system which extends the calculus $\mathbb{K}$ – by including the axiom schema $\mathsf{K} : \Box(A \to B) \to (\Box A \to \Box B)$ – and is modularly extended by instantiating $\mathcal{S}$.

Definition 1 (Proof system). *The ternary predicate $\mathcal{S}.\mathcal{H} \vdash A$, denoting the deducibility of a formula A from a set of hypotheses $\mathcal{H}$ in an axiomatic extension of logic K via schemas in the set $\mathcal{S}$, is inductively defined by the following conditions:*

[2] Notice that, although HOL Light lacks explicit mechanisms to support parametric and/or ad hoc polymorphism (unlike e.g. Isabelle/HOL), this distinction remains helpful in presenting the abstract structure of our (formalised) proof and discussing the potential portability of our results to proof assistants that implement this distinction through specific mechanisms.

- *For every instance A of axiom schemas for the calculus $\mathbb{K}$, $\mathcal{S}.\mathscr{H} \vdash A$;*
- *For every instance A of schemas in $\mathcal{S}$, $\mathcal{S}.\mathscr{H} \vdash A$;*
- *For every $A \in \mathscr{H}$, $\mathcal{S}.\mathscr{H} \vdash A$;*
- *If $\mathcal{S}.\mathscr{H} \vdash B \to A$ and $\mathcal{S}.\mathscr{H} \vdash B$, then $\mathcal{S}.\mathscr{H} \vdash A$;*
- *If $\mathcal{S}.\emptyset \vdash A$, then $\mathcal{S}.\mathscr{H} \vdash \Box A$ for any set of formulas $\mathscr{H}$.*

For that deducibility relation, we provide a formal proof of

Theorem 1 (Deduction theorem). *For any modal formulas A, B and any sets of formulas $\mathcal{S}, \mathscr{H}$, the following equivalence holds: $\mathcal{S}.\mathscr{H} \cup \{B\} \vdash A$ iff $\mathcal{S}.\mathscr{H} \vdash B \to A$.*

Moving to the semantics side, the current version of HOLMS contains the formalisation of basic notions of frames, relational models and validity therein via a forcing relation. Furthermore, we formalise the notion of validity in a class of frames.[3] Next, we formalise results from correspondence theory [32] collected in the table from Figure 1 by also defining a general predicate CHAR $\mathcal{S}$, representing the class of frames *characteristic* to the schema(s) $\mathcal{S}$. We leverage it for stating and proving

Theorem 2 (Soundness). *Let $\mathcal{S}$ be a set of axiom schemata. Let $\mathfrak{S}$ be the characteristic class of frames for $\mathcal{S}$. For every $A \in \mathbf{Form}_{\Box}$, if $\mathcal{S}.\emptyset \vdash A$ then $\mathfrak{S} \vDash A$.*

From those results, we can easily prove the consistency of the proof systems introduced by Def. 1.

2.3. Completeness theorem

The current version of HOLMS includes a refined formal proof of semantic completeness of any system $\mathbb{L} \in \{\mathbb{K}, \mathbb{T}, \mathbb{K}4, \mathbb{S}4, \mathbb{B}, \mathbb{S}5, \mathbb{GL}\}$. The modularity of the strategy from [22] lets us avoid code duplication in several parts of the implementation. The following ad-hoc polymorphic theorem then condenses the resulting completeness statement:

Theorem 3 (Completeness). *Let $\mathcal{S}$ be a set of axiom schemata. Let $\mathfrak{S}$ be the characteristic class of* finite *frames for $\mathcal{S}$. For every $A \in \mathbf{Form}_{\Box}$, if $\mathfrak{S} \vDash A$, then $\mathcal{S}.\emptyset \vdash A$.*

Proof Sketch. We proceed by contraposition, and prove that—given a generic modal formula A—*if $\mathcal{S}.\emptyset \nvdash A$, then $\mathfrak{S} \nvDash A$.*

This means that for each set of axioms $\mathcal{S}$ and for each modal formula A, we have to find a countermodel $\mathscr{M}_A^{\mathcal{S}}$ inhabiting $\mathfrak{S}$, and a 'counterworld' $w_A^{\mathcal{S}}$ inhabiting $\mathscr{M}_A^{\mathcal{S}}$ such that $w_A^{\mathcal{S}} \Vdash_{\mathscr{M}_A^{\mathcal{S}}} A$. To do so, we formalise the argument in [22, Ch. 5] and implement the following key strategy.

***Parametric part of the proof*:**

1. We identify a parametric notion of (counter)model $\mathscr{M}_A^{\mathcal{S}}$ in $\mathfrak{S}$ having: maximal consistent lists of modal formulas as possible worlds; an accessibility relation that verifies two given constraints; a valuation such that $V_A^{\mathcal{S}}(q, w)$ iff q is a subformula of A and $q \in w$. (GEN_STANDARD_MODEL)

2. We prove a *general truth-lemma* independent from the considered $\mathcal{S}$ and A. This step allows the reduction of the notion of *forcing* ($w \Vdash_{\mathscr{M}_A^{\mathcal{S}}} B$) to *membership* ($B \in w$). (GEN_TRUTH_LEMMA)

3. We identify a 'counterworld' X in $\mathscr{M}_A^{\mathcal{S}}$ such that $A \notin X$ and thus $X \nVdash_{\mathscr{M}_A^{\mathcal{S}}} A$. (GEN_COUNTERMODEL_ALT)

***Ad hoc polymorphic part of the proof*:**

I. For each system $\mathcal{S}$, identification of its specific countermodel $\mathscr{M}_A^{\mathcal{S}}$, and in particular of its accessi-

[3] We refer to our previous [23] for the details of the implementation.

bility relation $Rel_A^{\mathcal{S}}$;

II. Verification that this model satisfies the following two technical requirements for $Rel_A^{\mathcal{S}}$:

   a. $\langle W_{\mathcal{S}}^A, Rel_{\mathcal{S}}^A \rangle \in \mathfrak{C}$ ($\mathcal{S}$_MAXIMAL_CONSISTENT);

   b. for all $B \in \mathbf{Form}_{\Box}$, if $\Box B$ is a subformula of A then, for every $w \in W_{\mathcal{S}}^A$, $\Box B \in w$ if and only if, for all $x \in W_{\mathcal{S}}^A$, $wRel_{\mathcal{S}}^A x$ implies $B \in x$ ($\mathcal{S}$_ACCESSIBILITY_LEMMA).

□

As a technical aside, we notice that for step 2, we work on worlds that are lists of formulas without repetitions for purely practical reasons. Because of that choice, the formal proof of Theorem 3 establishes completeness w.r.t. frames sitting on the type-theoretic domain of formula lists. We overcome that annoying consequence of our implementation choice by applying a type-theoretic version of the bisimulation invariance lemma [33] (BISIMILAR_VALID ↗) which allows us to prove completeness w.r.t. any infinite domain – including formula sets commonly used in this kind of constructions – thanks also to the auxiliary lemma GEN_LEMMA_FOR_GEN_COMPLETENESS ↗.

3. Related and future work

By HOLMS, we aim to equip a proof assistant with a uniform and compositional mechanism for decision procedures for modal logics. To achieve that, the mechanisation of completeness results we have presented here is a key step, independently of the logic under consideration.

In our development, we have been inspired by independent works that individually present relevant aspects such as the natural extensibility of decision procedures of [34, 35, 36, 37, 38, 39] and the analogous generality of the formal proofs of completeness via canonical model constructions exhibited in [40, 41]. By merging those aspects in a uniform framework, we have managed to mechanise in a modular, uniform, and general way *six* central systems of the modal cube (K, T, K4, S4, B, S5) – already captured independently by those works – *plus* Gödel-Löb logic, which is one of the most important systems beyond the modal cube. At the current stage, the HOLMS library also contains a fully formalised uniform proof of adequacy for the *seven* normal modal systems w.r.t. relational semantics.[4]

A basic future task is thus to extend HOLMS to the *entire* modal cube [44] and further provability logics [45].

Another goal is to support multimodal languages by indexing modal operators – treating current monomodal systems as a special case – thereby significantly enhancing the expressiveness of the framework and broadening its potential applications in software and system verification [46, 47], [48, 49], [50], [51].

We also aim to integrate deep embeddings of enriched sequent calculi and their metatheory. That implementation would enable using HOLMS for interesting proof-theoretic investigations and facilitate integration with external derivation-search tools.

Another direction is generalising HOLMS to non-normal modal logics (with neighbourhood semantics) [52, 53, 54] and intuitionistic/constructive modalities [55, 56, 57], which are increasingly relevant in computer science. This extension would require adapting our methods and relaxing the current parametric approach to semantic adequacy.

More importantly, we plan to apply our modular mechanisation of the completeness theorem(s) to improve the performance of the (semi)decision procedures in HOLMS to cover more efficiently our proof-search mechanism for the systems we have discussed in the previous pages.

[4] A different methodology based on shallow embeddings of modal reasoning in HOL is documented in [42, 43].

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Declaration on generative AI

The authors have not employed any Generative AI tools.

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